Internal Note: Coordinate Systems of the Support Chamber

1 The object

For fixed $n \geq 1$, define the positive-indexed support row

$$T(n, k) = 1$$

if and only if

$$k = ab$$

for some integers

$$1 \leq a, b \leq n,$$

with

$$1 \leq k \leq n^2.$$

Thus, at level n, the support is first a one-dimensional object:

$$T(n, 1), T(n, 2), \ldots, T(n, n^2).$$

The support is naturally one-dimensional because it records the occurrence or non-occurrence of integers k.

2 Euclidean coordinates

Each integer k has a Euclidean decomposition relative to n:

$$k = qn + r, \qquad 0 \leq r < n.$$

Thus each position k determines a Euclidean coordinate

$$E_n(k) = (q, r).$$

This coordinate system is intrinsic to division by n. It is available at a single fixed level n. No motion or comparison between different values of n is required.

The Euclidean frame records the state of k relative to division by n.

3 Finite chamber coordinates

When the support row is arranged as an $n \times n$ bitmap, each entry also has a finite display position.

Let

$$F_n(k) = (u, v)$$

denote the finite chamber coordinate of the cell occupied by k in the $n \times n$ display.

We use the lower-left chamber corner as origin. Thus

$$0 \leq u < n, \qquad 0 \leq v < n,$$

where u increases to the right and v increases upward.

The important point is that F_n still belongs to level n. The coordinate pair (u, v) is not merely a point in an infinite plane; it is a point inside a particular finite square.

Thus the full information is really

$$(u, v; n).$$

The finite chamber coordinate remembers the box.

4 Cartesian coordinates

After a finite chamber coordinate system has been chosen, the same coordinate directions can be extended beyond the $n \times n$ box.

This gives ordinary Cartesian coordinates

$$(x, y) \in \mathbb{Z}^2,$$

or, if desired,

$$(x, y) \in \mathbb{R}^2.$$

These coordinates can describe points outside the current frame.

This is the key distinction:

$$(u, v; n)$$

is a coordinate inside the finite level-n chamber, while

$$(x, y)$$

is a coordinate in an extended plane.

The origin and orientation may be the same, but the Cartesian coordinate has forgotten the finite frame number n.

Thus Cartesian coordinates are not identical to finite chamber coordinates. They are the extension obtained after the finite frame has been detached from its level.

5 Coordinate transformations

Let

$$p = k - 1.$$

Here p is the zero-based linear index of the support row. Thus

$$0 \le p < n^2.$$

The one-dimensional support coordinate, Euclidean coordinate, and finite chamber coordinate are related as follows.

From	To	Formula
k	p	$p = k - 1$
p	k	$k = p + 1$
k	(q, r)	$q = \left\lfloor \dfrac{k}{n} \right\rfloor, \quad r \equiv k \pmod{n}, \quad 0 \le r < n$
(q, r)	k	$k = qn + r$
k	$(u, v; n)$	$u \equiv k - 1 \pmod{n}, \quad v = n - 1 - \left\lfloor \dfrac{k-1}{n} \right\rfloor$
$(u, v; n)$	k	$k = (n - 1 - v)n + u + 1$
p	$(u, v; n)$	$u \equiv p \pmod{n}, \quad v = n - 1 - \left\lfloor \dfrac{p}{n} \right\rfloor$
$(u, v; n)$	p	$p = (n - 1 - v)n + u$
(q, r)	$(u, v; n)$	$u \equiv qn + r - 1 \pmod{n}, \quad v = n - 1 - \left\lfloor \dfrac{qn + r - 1}{n} \right\rfloor$
$(u, v; n)$	(q, r)	$q = \left\lfloor \dfrac{(n - 1 - v)n + u + 1}{n} \right\rfloor, \quad r \equiv (n - 1 - v)n + u + 1 \pmod{n}$
$(u, v; n)$	(x, y)	$x = u, \quad y = v$
(x, y)	$(u, v; n)$	$u = x, \quad v = y, \quad 0 \le x, y < n$

The last line is only valid when (x, y) lies inside the level-n frame. Cartesian coordinates may describe points outside the frame, while $(u, v; n)$ may not.

Equivalently, the conversion between Euclidean and finite chamber coordinates may be written piecewise.

Since

$$k = qn + r,$$

we have

$$u \equiv r - 1 \pmod{n}.$$

Moreover,

$$v = \begin{cases} n - 1 - q, & r \neq 0, \\ n - q, & r = 0. \end{cases}$$

Conversely, from

$$k = (n - 1 - v)n + u + 1,$$

we get

$$r \equiv u + 1 \pmod{n},$$

and

$$q = \begin{cases} n - 1 - v, & u \neq n - 1, \\ n - v, & u = n - 1. \end{cases}$$

The exceptional cases arise only from the positive indexing of k and the fact that multiples of n have remainder 0.

6 The S-frame

The S-frame is subtler.

It is not merely the use of Cartesian coordinates. The S-frame is the distinguished Cartesian-style frame whose origin and orientation are identified through frame-to-frame comparison:

$$n \mapsto n + 1.$$

At one fixed level n, the Euclidean coordinates are immediately defined. But the S-frame is not fully revealed by a single still image.

It emerges when the support chambers are compared across levels and one observes relative constancy, persistence, rays, boundaries, and stable motion around a distinguished point.

This point is the S-point.

Thus

$$E_n(k) = (q, r)$$

is a static arithmetic coordinate, while the S-frame is a dynamically identified coordinate frame.

The S-frame is therefore not merely Cartesian. It is a Cartesian-style frame whose origin has been selected by the motion of the support itself.

7 Recorded properties of the support

The following list is not intended to be exhaustive. It records the properties presently used or observed in our study of the support.

1. **Elementary definition.** For fixed $n \geq 1$, the support is defined by

$$T(n,k) = 1$$

if and only if

$$k = ab$$

for some

$$1 \leq a, b \leq n.$$

Otherwise,

$$T(n,k) = 0.$$

2. **Bounded ambient range.** At level n, the support is defined on the full interval

$$1 \leq k \leq n^2.$$

3. **Self-indexing.** Each position k refers to the integer k itself. Thus the sequence records occurrence or non-occurrence at the integer being indexed.

4. **Zeros have structural meaning.** A zero at position k means that k has no divisor pair

$$k = ab$$

with

$$1 \leq a, b \leq n.$$

Thus zeros encode missing divisor pairs inside the bounded $n \times n$ square.

5. **Divisor-window formulation.** Equivalently,

$$T(n,k) = 1$$

if and only if k has a divisor d such that

$$1 \leq d \leq n$$

and

$$1 \leq \frac{k}{d} \leq n.$$

6. **Relation to the multiplication table.** The support records exactly the set of distinct entries occurring in the $n \times n$ multiplication table

$$\{ab : 1 \leq a, b \leq n\}.$$

7. **Relation to multiplicity.** If

$$M(n,k) = \#\{(a,b) : 1 \leq a, b \leq n,\ ab = k\},$$

then

$$T(n,k) = 1$$

if and only if

$$M(n,k) > 0.$$

8. **Relation to the distinct products problem.** The row sum

$$\sum_{k=1}^{n^2} T(n,k)$$

is the number of distinct products in the $n \times n$ multiplication table.

9. **Representation-invariant flat form.** In flat sequence form, the support does not list the attained products in an arbitrary order. It records the ambient yes-or-no occurrence structure across the full interval

$$1, \ldots, n^2.$$

10. **Quadratic block structure.** The n-th support row has length

$$n^2.$$

Thus the rows are naturally organized into quadratic blocks.

11. **Natural framing.** At level n, the support has a distinguished finite frame of size

$$n \times n.$$

This frame is not added externally; it is inherited from the length n^2 and the multiplication table bounds.

12. **Boundary values.** For each $n \geq 1$,

$$T(n,k) = 1$$

for

$$1 \leq k \leq n,$$

because

$$k = 1 \cdot k.$$

Also,

$$T(n,n^2) = 1,$$

because

$$n^2 = n \cdot n.$$

13. **One-dimensional support.** For fixed n, the support is naturally a one-dimensional row:

$$T(n,1), T(n,2), \ldots, T(n,n^2).$$

14. **Euclidean two-dimensional reading.** Each integer k admits a Euclidean decomposition

$$k = qn + r, \qquad 0 \leq r < n.$$

Thus the same support entry has a natural two-dimensional Euclidean state

$$(q,r).$$

15. **Finite chamber reading.** The row of length n^2 can be arranged as an $n \times n$ bitmap. This gives finite chamber coordinates

$$(u,v;n),$$

with

$$0 \leq u,v < n.$$

16. **Coexistence of flat and two-dimensional structure.** The support is simultaneously a flat yes-or-no sequence in k and a structured $n \times n$ chamber under its Euclidean or frame reading.

17. **Inherited Cartesian structure.** The $n \times n$ display retains Cartesian structure inherited from the multiplication table and from the quotient-remainder decomposition.

18. **Nontrivial visible structure.** In square-grid displays, the support is not visually featureless. It exhibits visible internal structure, including rays, bands, regions, boundary effects, and persistent forms.

19. **Sensitivity to n.** The chamber can change visibly with the level n. Nearby levels, such as n and $n + 1$, may display related but nonidentical chamber structures.

20. **Persistence across levels.** When chambers are compared under

$$n \mapsto n + 1,$$

some structures remain recognizable across levels.

21. **Transport law for fixed products.** For a fixed product

$$k = ij = qn + r$$

at level n, its remainder at level $n + 1$ satisfies

$$R \equiv r - q \pmod{n + 1}.$$

This gives a frame-to-frame transport rule for the Euclidean state of the same product.

22. **Static and dynamic coordinates.** The Euclidean coordinate

$$(q, r)$$

is defined at a single level n. The S-frame is identified through comparison of successive levels.

23. **S-point observation.** In the chamber orientation used here, a distinguished point or region, denoted S, appears as a natural anchor for frame-to-frame comparison.

24. **Relative constancy in the S-frame.** When viewed relative to the S-point, some chamber features show reduced apparent displacement or repeated alignment across levels.

25. **Delta observations.** Frame-to-frame differences contain visible $\Delta = 0$, near-$\Delta = 0$, and $\Delta = 1$-type phenomena. These are observed as structured events rather than uniformly scattered changes.

26. **Predictive use of chamber structure.** Certain rays, bands, or boundary-related structures identified in one frame have been used to anticipate corresponding features in later frames.

27. **Memory-stride generation.** The chamber is generated by writing multiplicative hits into a one-dimensional memory row and then reading the row with stride n:

```
import numpy as np

row = np.zeros(n*n, dtype=np.uint8)
for a in range(1, n+1):
    for b in range(1, n+1):
        row[a*b - 1] = 1
grid = row.reshape(n, n)
```

Thus the two-dimensional chamber is obtained by a stride-n reading of the one-dimensional support row.

28. **Low descriptive complexity.** The definition of the support is elementary, while the resulting chamber displays visible internal structure.

29. **Relation to the master array.** If

$$A(n,q,r) = \#\{(a,b) : 1 \leq a,b \leq n,\ ab = qn + r\},$$

then

$$T(n,qn+r) = 1$$

if and only if

$$A(n,q,r) > 0.$$

30. **Support as the shadow of multiplicity.** The support forgets multiplicity but preserves occurrence. It is therefore the Boolean shadow of the multiplicity distribution.

31. **Universality of the construction.** The construction depends only on multiplication, bounded factor pairs, and Euclidean division. No auxiliary choice of weights, ordering of products, smoothing, or numerical approximation is involved.

32. **Rare coexistence of properties.** The support combines an elementary definition, self-indexing flat form, quadratic framing, divisor information, multiplication-table structure, Euclidean coordinates, finite chamber coordinates, visible internal geometry, and frame-to-frame transport behavior.

8 Delta observations

Let

$$\Delta_n$$

denote the frame-to-frame change observed when passing from level n to level $n+1$.

By relative constancy we mean only that certain features remain recognizable across levels when the chambers are compared relative to the distinguished S-point or S-frame.

In the Euclidean frame, the transport of a fixed product

$$k = ij$$

is governed by

$$R \equiv r - q \pmod{n+1}.$$

Thus the same integer or product does not merely receive a new drawing at level $n+1$. It receives a new Euclidean state.

In the finite chamber and S-frame readings, the comparison

$$n \mapsto n+1$$

reveals several observed forms of relative constancy.

First, there are cells or regions whose chamber position changes little, or not at all, when viewed relative to the S-point. These are observed as $\Delta = 0$ or near-$\Delta = 0$ features.

Second, there are visible rays emanating from the S-region. These rays remain recognizable across levels, even though the finite frame itself grows.

Third, there are frame-to-frame changes of small displacement, especially $\Delta = 1$-type events. These appear in bands or families rather than as isolated accidents.

Fourth, some ray-like or boundary-related structures have shown predictive power: once a structure is identified in the S-frame, its continuation into later frames can often be anticipated before the next chamber is drawn.

The role of the S-frame is therefore observationally significant. It does not merely redraw the support. It exposes relative persistence under growth.

In compact form:

$$\text{Euclidean transport governs state change,}$$

while

$$\text{the } S\text{-frame reveals constancy under frame growth.}$$

This note does not assert a general theorem for all such delta structures. It only records the observed fact that the S-frame was identified through frame-to-frame constancy and has already been useful for recognizing and predicting persistent chamber features.

9 Summary of the layers

The support carries several coordinate readings.

First,

$$k$$

is the one-dimensional integer position.

Second,

$$(q, r)$$

is the Euclidean coordinate relative to division by n.

Third,

$$(u, v; n)$$

is the finite frame coordinate inside the $n \times n$ chamber.

Fourth,

$$(x, y)$$

is the extended Cartesian coordinate after the finite frame has been detached.

Finally, the S-frame is the special Cartesian-style frame whose origin is discovered through persistence under the transition

$$n \mapsto n + 1.$$

So the hierarchy is

$$\boxed{k}$$

as one-dimensional support position,

$$\boxed{(q, r)}$$

as Euclidean state at level n,

$$\boxed{(u, v; n)}$$

as finite chamber coordinate,

$$\boxed{(x, y)}$$

as extended Cartesian coordinate, and

$$\boxed{S}$$

as motion-discovered frame of relative constancy.

10 Observation

The support is unusual because these coordinate systems are not arbitrary decorations.

The one-dimensional coordinate k comes from the definition of support.

The Euclidean coordinate (q, r) comes from division by n.

The finite frame coordinate $(u, v; n)$ comes from arranging the support row into its natural $n \times n$ chamber.

The extended Cartesian coordinate (x, y) arises when the frame is treated as part of a larger plane.

The S-frame is identified only through motion across levels.

Thus the support does not merely admit many drawings. It carries a sequence of increasingly geometric readings, each arising from a different structural aspect of the same object.

A compact formulation is:

$$\text{division defines } E, \qquad \text{display defines } F, \qquad \text{extension defines } (x, y), \qquad \text{motion selects } S.$$

Equivalently:

$$\boxed{\text{The Euclidean frame is defined by division.}}$$

$$\boxed{\text{The finite frame is defined by display.}}$$

$$\boxed{\text{The Cartesian plane is obtained by extending the frame.}}$$

$$\boxed{\text{The } S\text{-frame is selected by motion.}}$$

The same support object passes through all four readings without changing its underlying definition.